Rutgers University: Algebra Written Qualifying Exam

January 2009: Day 1 Problem 3 Solution

Exercise. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing 56. (You may quote the Sylow theorems without proof.)

Solution.

Let G be a group of order 56.

$$56 = 2^3 \cdot 7$$

By the third Sylow theorem,

$$n_7 \equiv 1 \mod 7 \quad \text{and} \quad n_7 \mid 8 \quad \Longrightarrow \quad n_7 = 1 \text{ or } 8$$
$$n_2 \equiv 1 \mod 2 \quad \text{and} \quad n_2 \mid 7 \quad \Longrightarrow \quad n_2 = 1 \text{ or } 7$$

If G does *not* have a Sylow p-subgroup that is normal then, by the second Sylow theorem,

$$n_7 = 8 \qquad \text{and} \qquad n_2 = 7$$

If $n_7 = 9$, G has $8(7-1) = 48$ elements of order 7.

There are $56 - 48 = 8$ remaining elements in G

This is a single Sylow 2-subgroup

$\Longrightarrow$ there must be only one Sylow 2-subgroup

$\Longrightarrow$ the Sylow 2 subgroup is normal in G by the second Sylow theorem